The Ant and the Turntable - Frames of Reference

$$S = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Figuring out spiral lengths requires a bit of calculus. In the previous problem, we saw that the arc length integral can be written in polar coordinates where the function, y = F(x) is replaced by the polar function **r(θ).**

Because this formula is completely general, the variable, θ, can refer to angle, time or any other independent variable which leads to an arc-like geometry in the dependent variable defined by r(θ). Here is an interesting application.

An ant takes a journey from the center of a CDrom (r=0) to a point at its edge (r=5cm) at a leisurely pace of 5 cm/minute. Meanwhile, the turntable is spinning at 1 rotation per minute.

Problem 1 - Draw a sketch, to scale, of the turntable that shows the ant's motion from its own, stationary, perspective.

Problem 2 - Draw a free-hand sketch, to scale, of the turntable that shows the ant's motion from the perspective of an outside, stationary observer.

Problem 3 - What is the equation that describes the radial motion of the ant with respect to time?

Problem 4 - How far did the ant travel in the radial direction in A) 15 seconds?, B) 60 seconds?

Problem 5 - Evaluate the arc length integral formula for S(t) to determine the length of the ant's arc between r=0 and r = 5 centimeters.

Inquiry: Explain A) how it can be that there are two path lengths and travel times in the problem, but there is only one ant? and B) which of the two answers is correct?
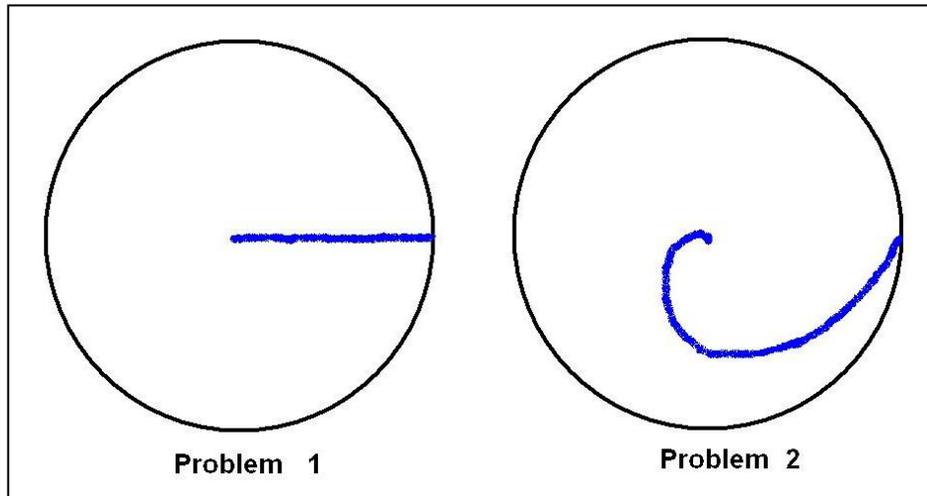
Problem **1**   Problem **2**

Problem 3**)**   **R(T) = (5 cm/minute) T**   where T is given in minutes of time.

Problem 4)   A) 0.25 minutes x 5 cm/minute = 1.25 centimeters,
                 B) 1 minute x 5 cm/minute **= 5 centimeters.**

Problem 5)   dr/dT= 5 cm/m so the integrand becomes by substitution, dS = (r^2 + (dr/dt)^2)$^{1/2}$ or
dS = ((5 T)^2 + 5^2)$^{1/2}$ dT . This simplifies to dS = 5 (1 + T^2)$^{1/2}$ dT . The integral then becomes,

$$S(t) = 5 \int_{0}^{t} \sqrt{1 + T^2} \ dT$$

This can be integrated by using the substitution T = sinh(x) where dT = cosh(x) dx so that
(1 + T^2)$^{1/2}$ dT becomes cosh(x) cosh(x) dx, or cosh^2(x)dx. The integral is a fundamental
form whose solution is   sinh(2x)/4 + x/2 + C. Evaluating from T =0 to T=t one gets the
general arc length formula for the ant of S(t) = 5 cm/m x (sinh(2t)/4 + t/2) where t is the
elapsed time expressed in minutes.

To travel from r=0 to r=5cm, the ant takes 1 minute, so S(1) = 5 cm/minute x (3.63/4 + 1/2) = 5
cm/m (1.4 minutes) = 7.0 centimeters.

Inquiry: There are two reference frames, and the movement of the ant will appear different
to the two observers. Only in the 'proper' frame of reference locked to the ant will the
movement of the ant appear to be 'simple' as it travels in a straight line from the center to the
edge of the CDROM according to R(T) = 5 T. In all other 'improper' reference frames, the
motion will appear more complicated, and trace out more complex paths because of the
action of fictitious forces. For example, if you are sitting in a moving car on the freeway and
you drop a ball to the floor, from your perspective, it travels 1/2 meter straight down. For
someone standing by the side of the road, the ball travels in a long 100-meter arc.

B) They are both correct, but they apply to two different frames of reference who are seeing
the movement differently because of their relative state of motion.